Correction of Wavefront Aberration and Communication using Aperture Synthesis

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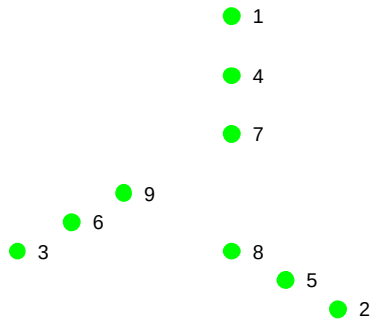
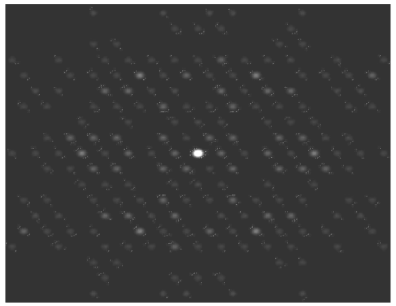
Background



- Interferogram formed by aperture array
  - Aperture pairs, e.g. (*i*,*k*), generate fringes
  - Fringe frequency proportional to spacing between apertures (*i*,*k*)
- By van Cittert-Zernike theorem, fringe properties give information about an object in the frequency domain
  - Aperture spacings sample object's complex frequency representation



- Aberrations affect aperture phases
  - Sampling ⇒ spacing and object phases added
  - Resultant appears as aberrated object information
- Correction of aperture phases sought
  - Phase closure
  - Redundant Spacings Calibration (RSC)

Outline

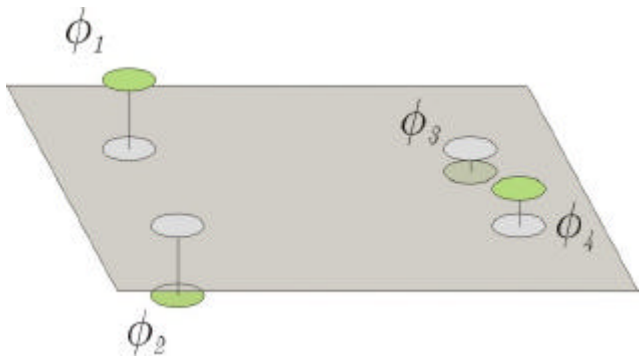
- Two methodologies using principles of RSC
  - Calibration/correction aperture phase
    - Regulator control
    - Image sharpness criterion
  - Direct identification of aperture and object phases
    - Object reconstruction
  - Both methods applicable to AO

- Free Space Optical Communications (FSOC)
  - Thin screen ‘atmosphere’ modelling
    - Intensity function characteristics for optimising collector arrays
  - Using 2nd phase screen

Image Sharpness RSC

- Active phasing of apertures corrects wavefront curvature by altering ‘effective’ aperture phases
  - e.g. 4 aperture array

$$f_i = \bar{f}_i - \hat{f}_i \qquad i = 1,2,\ldots N$$



  - $\bar{f}_i$ = true aberration phase
  - $\hat{f}_i$ = estimated correction phase
- Aperture spacings sampling the same spatial frequencies $\Rightarrow$ same effective phase difference
  - Results in $\leq (N-3)$ independent ‘redundant’ conditions which must be satisfied.
    - Depends on array configuration
  - If this fails to hold (aberration), auto-correlation magnitude $\Rightarrow$ image sharpness is reduced

Image Sharpness RSC

- Further 3 a priori conditions required for unique solution
  - Constraining two estimated phase differences to zero imposes a tilt plane on OTF phase $\Rightarrow$ image steering

$$\hat{f}_i - \hat{f}_k = 0 \qquad \hat{f}_l - \hat{f}_m = 0$$

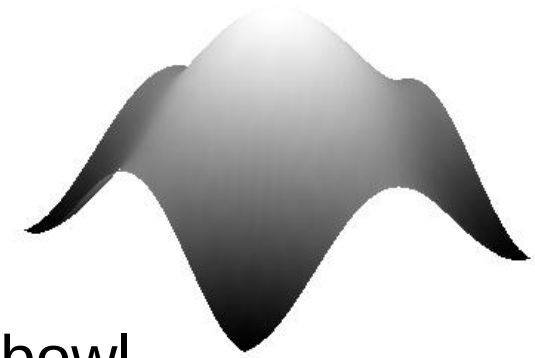
    - Two phase differences must be on nonparallel spacings
  - Apply arbitrary offset to a single estimated phase
  - N - square system of equations in the estimated aperture phases formed from these conditions - example

$$\hat{f}_1 = 0$$
$$\hat{f}_1 - \hat{f}_2 = 0$$
$$\hat{f}_1 - \hat{f}_3 = 0$$
$$\hat{f}_1 - \hat{f}_2 - \hat{f}_3 + \hat{f}_4 = \bar{f}_1 - \bar{f}_2 - \bar{f}_3 + \bar{f}_4$$

- All coefficient are integers; constants and variables are mod 2*p*
- Determinant of this system has great significance for the solution

Image Sharpness RSC

- When redundant conditions met, auto-correlation magnitude maximised
  - Parseval's theorem implies image sharpness can be used as evaluation criterion
- If apertures are uniformly illuminated, the sharpness optimisation surface is a $\cos^2(\cdot)$ bowl in mod $2p \subset \mathbb{R}^N$.
  - Maximum can be located in a single step



- If untrue, surface is no longer a simple bowl
  - Unknown aperture illumination $\Rightarrow$ unknown surface
  - Should remain monotone increasing towards a single maximum
  - Iterative solution available

Phase Identification RSC

- An interferogram is formed by the aperture array
  - Each fringe set generated by each aperture pair e.g. (*i*,*k*)
  - Fringe frequency $\propto$ spacing between apertures
  - Phase and visibility measured by taking Fourier transform
- Phase relations of each fringe set/aperture pair can be written as

$$m_{ik} = q_{ik} + f_i - f_k$$

and formed into a system of equations rank deficient by N.

- Aperture spacings sampling the same spatial frequencies $\Rightarrow$ same object phase observed
  - $(N-3)$ independent 'redundant' conditions necessary for system to be made full rank
    - Array design must satisfy this condition

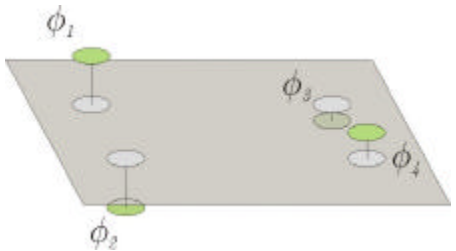
Phase Identification RSC

- Remaining rank deficiency of 3 addressed by a priori object independent information
  - Two a.p. conditions will apply a tilt to the object phases
    - Mislocates object reconstruction, but only morphology important.
  - Final condition sets a reference level for aperture phases.

$$
\left[\begin{array}{cccccc:cccc}
1 & 0 & & \cdots & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & & \vdots & 0 & 1 & 0 & -1 & 0 \\
 & & 1 & & & & 1 & 0 & 0 & -1 \\
\vdots & & & 1 & 0 & \vdots & 0 & 1 & -1 & 0 \\
0 & \cdots & & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & & \cdots & 0 & 1 & 0 & 0 & 1 & -1 \\
\hdashline
0 & \cdots & & & & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & & \cdots & & 0 & 0 & \cdots & & 0 \\
0 & 1 & 0 & \cdots & & 0 & \vdots & \ddots & & \vdots \\
1 & 0 & \cdots & & & -1 & & & & \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & \cdots & & 0
\end{array}\right]
\begin{bmatrix} \theta_{12} \\ \theta_{13} \\ \theta_{14} \\ \theta_{23} \\ \theta_{24} \\ \theta_{34} \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}
=
\begin{bmatrix} m_{12} \\ m_{13} \\ m_{14} \\ m_{23} \\ m_{24} \\ m_{34} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Phase Identification RSC

- Row operations can be performed on resulting matrix to make the system square
  - Zero valued equations removed
  - Corresponding matrix *D*-block is N-square and identical to matrix used in image sharpness earlier $\Rightarrow$ same determinant
- Aperture phase estimates depend only on measured phase values $\Rightarrow$ calculated independent of object phases



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ m_{34} - m_{12} - m_{13} + m_{24} \end{bmatrix}$$

  - Object phases q_{ik} then calculable to give object reconstruction

Modulo 2π Arithmetic

- Modulo *2p* nature of phases causes a problem with both methods of correction
- In terms of image sharpness
  - Triangularising the matrix reveals it's determinant
    - All values will be integers
    - Example here for 4-aperture parallelogram

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \hat{f}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{f}_1 - \bar{f}_2 - \bar{f}_3 + \bar{f}_4 \\ 0 \\ 0 \end{bmatrix} \qquad \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} = -2$$

  - Appearance of any number ? ±*1* on the leading diagonal means multiple solutions exist, e.g. with a *2*

$$\bar{f}_1 - \bar{f}_3 + \hat{f}_2 = \bar{f}_2 - \bar{f}_4 - \hat{f}_2$$

Modulo 2π Arithmetic

- With phase identification
  - *D*-block identical to the matrix constructed in the image sharpness method
  - Independent system for aperture phases
  - Solving yields the form

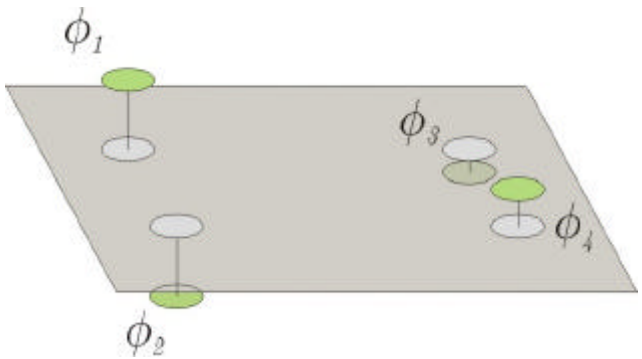
$$f_i = \sum \overline{m}_{lm} \Big/ \det D = \left(\sum m_{lm} + n2p \right) \Big/ \det D$$

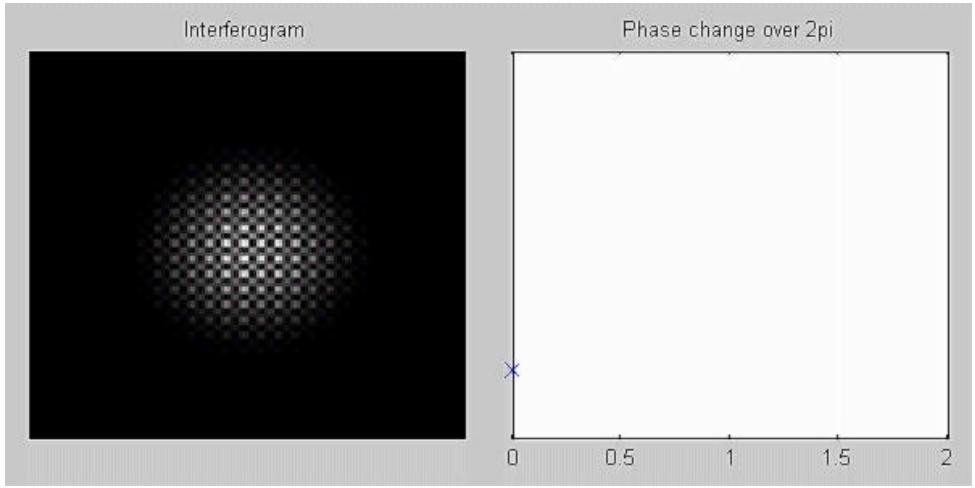
  - Result may be ambiguous by multiples of π
- Low determinant (e.g. ± 1) associated with poorer system conditioning
  - generally

$$k(A) = \left\| A^{-1} \right\| \left\| A \right\| = \left\| \operatorname{adj} A \right\| \left\| A \right\| \Big/ \det A$$

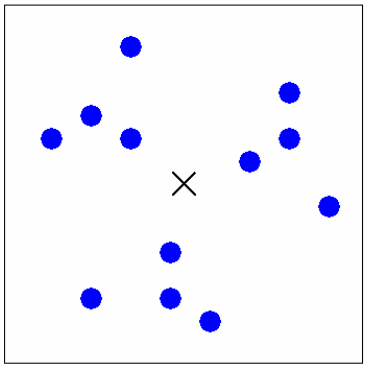
RSC Demonstrations

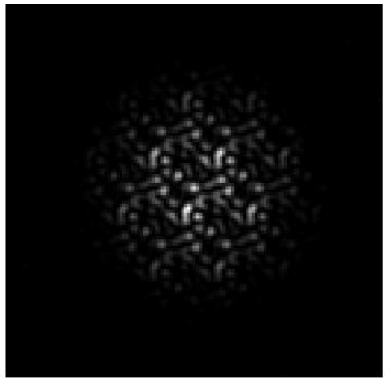
- Image sharpness

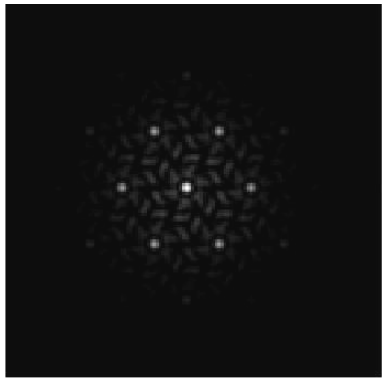




- Phase identification
  - Correction of aberrated PSF – note shift due to tilt.



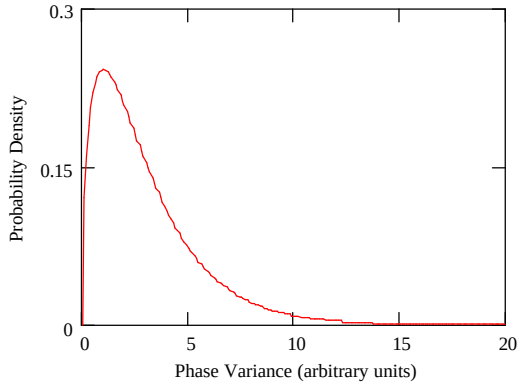


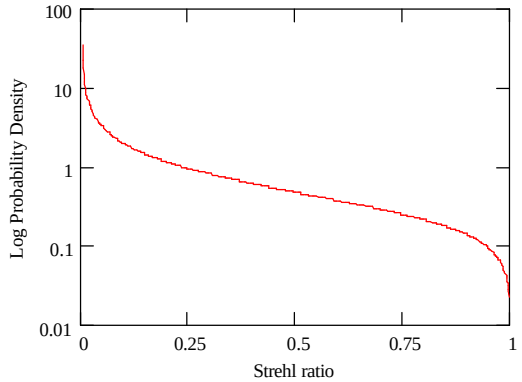


Lucky Imaging

- Filled aperture case
  - Low aberration variance across an aperture sought, but small probability
    - Low variance $\Rightarrow$ Strehl ratio, $S \sim 1$ $\Rightarrow$ relatively undistorted images.
  - Density function of S calculable using Marechal approximation (subject to assumptions).

$$S \approx e^{-s_f^2}$$





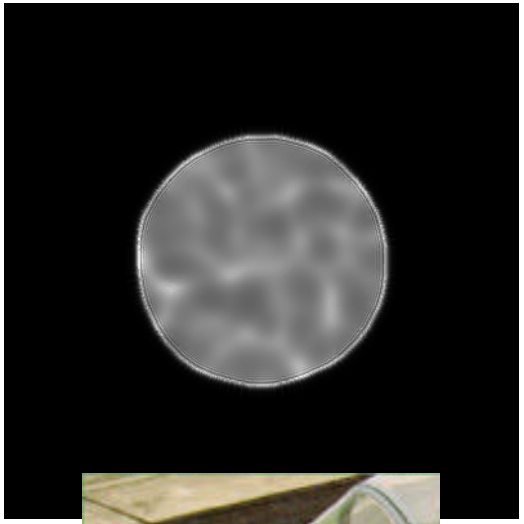
  - Many snapshots necessary to find a 'lucky' few.
    - Depends on size of telescope relative to correlation length
    - Image sharpness criterion allows automatic search.

Lucky Imaging

- Synthetic aperture case
  - Aperture array takes N samples of the wavefront
  - Aperture size influences complexity of phase modes passed
    - δ - function samples only piston
    - Real apertures see degrees of tilt and higher modes
    - Typically tilt is the highest significant order
  - Probability of lucky snapshot depends on N
    - Joint density function of samples
    - Sufficient frequency content sampled $\Rightarrow$ smaller N , higher probability of being lucky
  - Image sharpness solution must be met by the samples
    - Piston only array with redundancies $\Rightarrow$ RSC solution
  - Tilt case more complicated – less chance of being ‘phased’
  - L.I. analogous to Free Space Optical Communications
    - Array designs for imaging: low probability of lucky snapshot
    - FSOC aims to optimise array configuration to increase probability

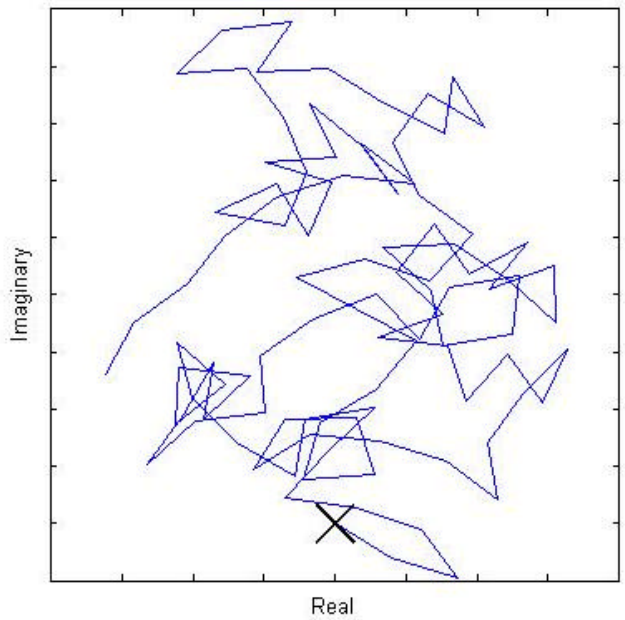
Communications

- Background
  - Free Space Optical Communications using lasers
  - Characteristics of irradiance field vary with time and propagation distance
    - Caustic and speckle formation in the intensity function
  - Large collector required for optimum signal continuity
  - Investigating use of arrays of smaller collectors
    - Airborne platforms requirements
    - Understanding of intensity properties sought
    - Different at different propagation distances



Communications

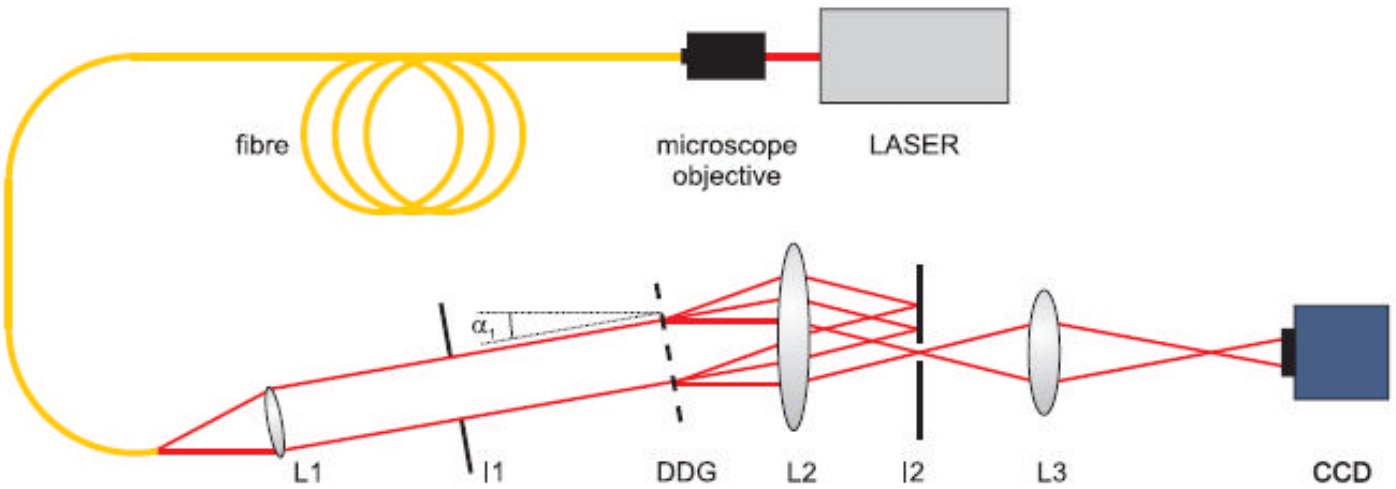
- Theoretical Investigation
  - Analytical approach
    - Based on propagation theory
    - Calculations accurate but unnecessarily unwieldy
  - Phenomenological approach
    - Scattering centres in source field
    - Each point in the target field results from a finite complex summation
    - Probabilistic treatment of scattering centres modelled using ‘bunching’ statistics
    - Closed form expressions for point statistics, e.g. *K*-distribution



$$\sum a_i e^{jy_i}$$

Communications

- Laboratory Experiments
  - Phase functions generated in ±1 orders of detour phase distorted diffraction grating (DDG)
  - Capability
    - Analogue computation of intensity function at almost arbitrary propagation distances by axial positioning of lens L3
  - Limitations
    - Phase screen thin and static

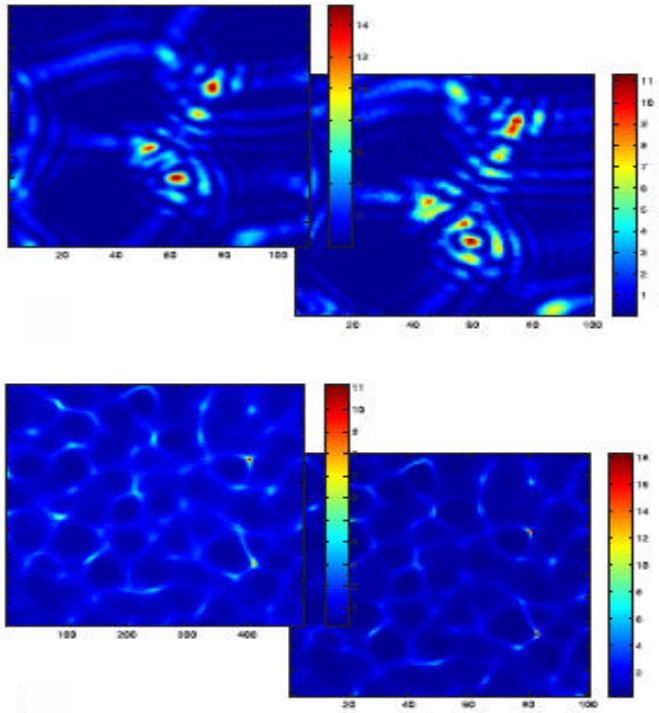


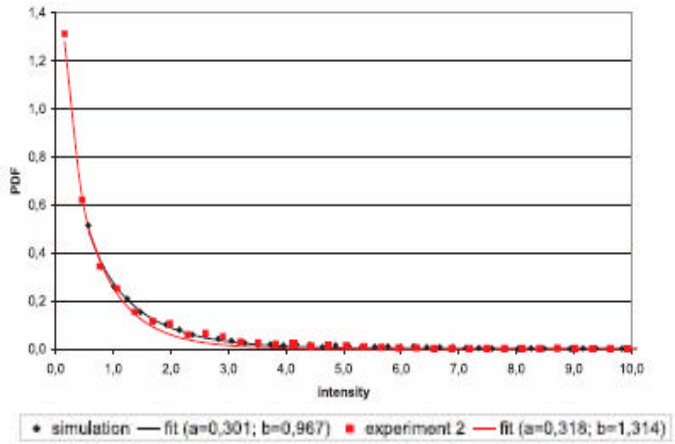
Communications

- Computer Simulations
  - Propagation by angular spectrum of plane waves
  - Capabilities
    - Set of phase functions pseudo-denumerable
    - Dynamic phase functions
    - Ensemble simulations
  - Limitations
    - Thin screen condition remains
    - Single wavelength and limited propagation distance
- Modelling Goals
  - Inductive validation of experimental and computer simulation
  - Validation and qualification of theory
    - Intensity point and spatial statistics
    - Use of theory for FSOC array optimisation

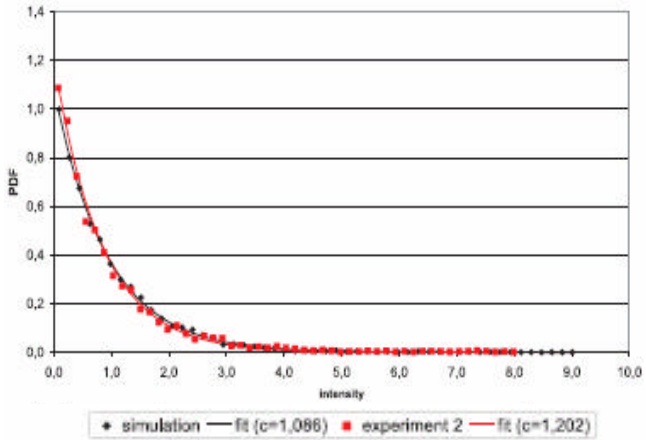
Communications

- Validation of lab/computer simulation
  - Left: experimental
  - Right: simulation





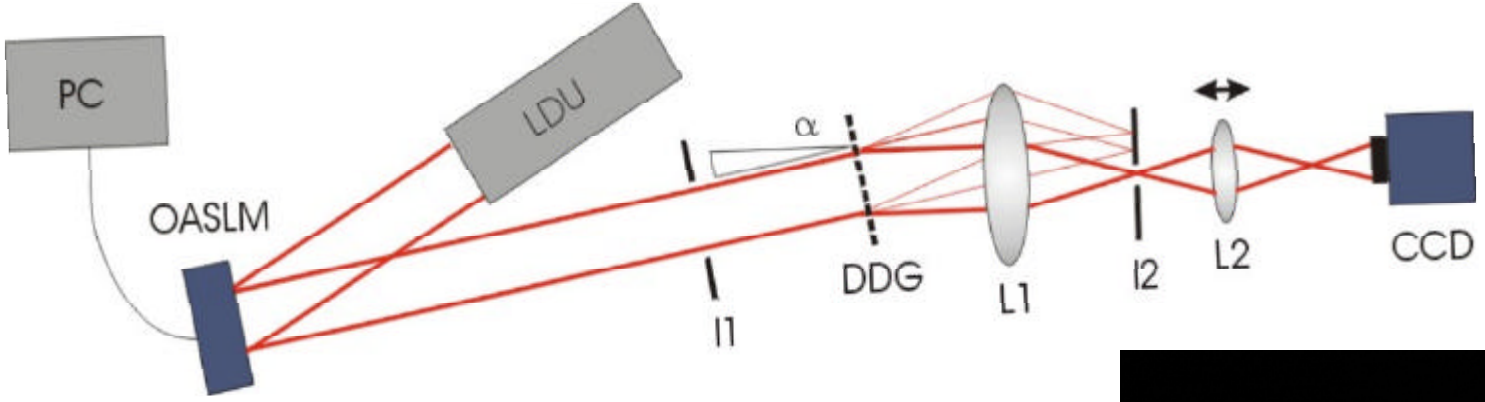
K-distribution
(near field)



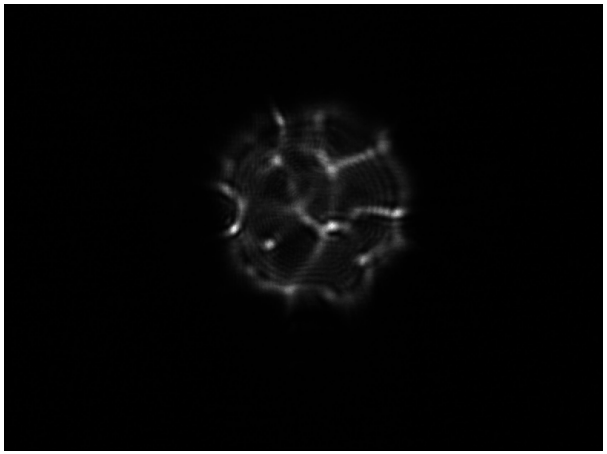
Negative Exponential
(far field)

2nd Phase Screen

- Modelling propagation through an inhomogeneous medium using a second phase screen
  - OASLM generated phase function propagated onto DDG
    - Up to 1? phase change

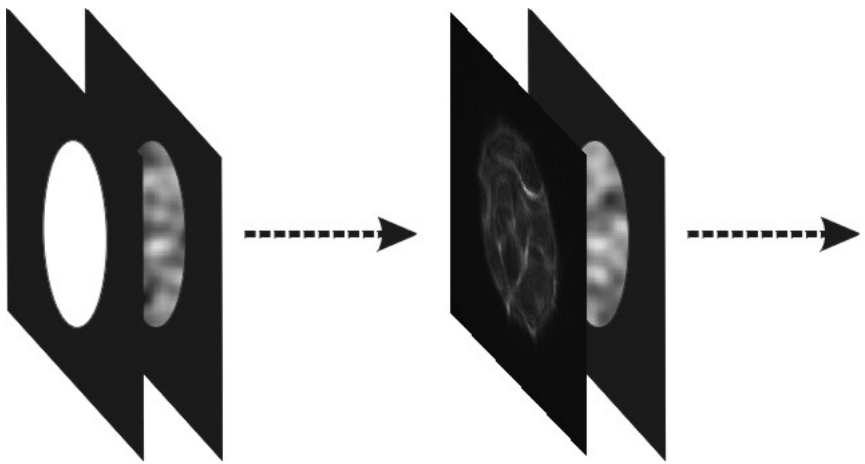


- No single phase-only screen can model multiple phase screen propagation
- Allows anisoplanatism to be modelled
- Moving phase patterns possible



2nd Phase Screen

- Intensity statistics relevant to FSOC particularly interesting
  - Rate of emergence and influence of regions distinguishable by intensity function characteristics
    - Focussing/caustics – near field
    - Speckle – far field
  - Influence of the first screen on the second

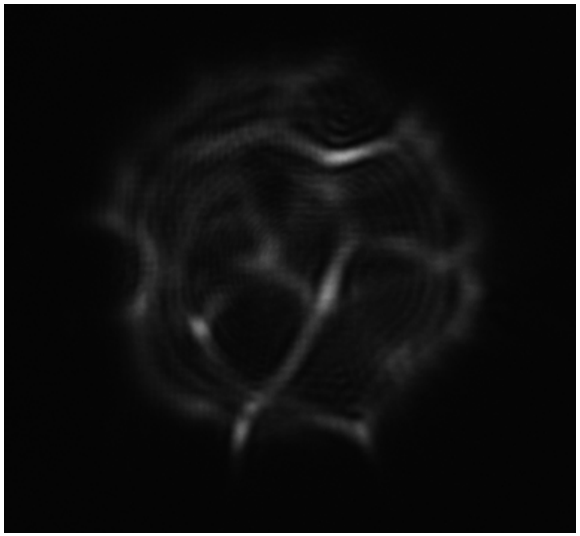


- Amplitude distribution of the first ‘samples’ regions of the second
- Intensity function due to both is naïve of some regions in the second

Array Design

- Imaging
  - Spatial frequency coverage matching object content important
  - Spacings must include sufficient redundancy for RSC solutions
    - Designs must allow unit determinant
  - Aperture sizes to limit wavefront modes passed

- Communications
  - Array configuration must be optimised to maximise probability of at least one aperture seeing high intensity
    - Intensity characteristics depend on atmosphere and propagation distance
  - Platform limits aperture sizes



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