Correction of Wavefront Aberration and Communication using Aperture Synthesis

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Background

- Interferogram formed by aperture array
  - Aperture pairs, e.g. \((i,k)\), generate fringes
  - Fringe frequency proportional to spacing between apertures \((i,k)\)
- By van Cittert-Zernike theorem, fringe properties give information about an object in the frequency domain
  - Aperture spacings sample object’s complex frequency representation
- Aberrations affect aperture phases
  - Sampling \(\Rightarrow\) spacing and object phases added
  - Resultant appears as aberrated object information
- Correction of aperture phases sought
  - Phase closure
  - Redundant Spacings Calibration (RSC)
Outline

• Two methodologies using principles of RSC
  – Calibration/correction aperture phase
    • Regulator control
    • Image sharpness criterion
  – Direct identification of aperture and object phases
    • Object reconstruction
  – Both methods applicable to AO

• Free Space Optical Communications (FSOC)
  – Thin screen ‘atmosphere’ modelling
    • Intensity function characteristics for optimising collector arrays
  – Using 2\textsuperscript{nd} phase screen
• Active phasing of apertures corrects wavefront curvature by altering ‘effective’ aperture phases
  – e.g. 4 aperture array
  \[ \phi_i = \bar{\phi}_i - \hat{\phi}_i \quad i = 1, 2, \ldots N \]
  – \( \bar{\phi}_i \) = true aberration phase
  – \( \hat{\phi}_i \) = estimated correction phase

• Aperture spacings sampling the same spatial frequencies \( \Rightarrow \) same effective phase difference
  – Results in \( \leq (N - 3) \) independent ‘redundant’ conditions which must be satisfied.
    • Depends on array configuration
  – If this fails to hold (aberration), auto-correlation magnitude \( \Rightarrow \) image sharpness is reduced
• Further 3 a priori conditions required for unique solution
  – Constraining two estimated phase differences to zero imposes a tilt plane on OTF phase \( \Rightarrow \) image steering
    \[
    \hat{\phi}_i - \hat{\phi}_k = 0 \quad \hat{\phi}_l - \hat{\phi}_m = 0
    \]
  • Two phase differences must be on nonparallel spacings
  – Apply arbitrary offset to a single estimated phase
  – \( N \)-square system of equations in the estimated aperture phases formed from these conditions - example
    \[
    \begin{align*}
    \hat{\phi}_1 &= 0 \\
    \hat{\phi}_1 - \hat{\phi}_2 &= 0 \\
    \hat{\phi}_1 - \hat{\phi}_3 &= 0 \\
    \hat{\phi}_1 - \hat{\phi}_2 - \hat{\phi}_3 + \hat{\phi}_4 &= \bar{\phi}_1 - \bar{\phi}_2 - \bar{\phi}_3 + \bar{\phi}_4
    \end{align*}
    \]
    • All coefficient are integers; constants and variables are mod \( 2p \)
    • Determinant of this system has great significance for the solution
• When redundant conditions met, auto-correlation magnitude maximised
  – Parseval’s theorem implies image sharpness can be used as evaluation criterion
• If apertures are uniformly illuminated, the sharpness optimisation surface is a \( \cos^2(\cdot) \) bowl in mod \( 2p \subset \mathbb{R}^N \).
  – Maximum can be located in a single step
• If untrue, surface is no longer a simple bowl
  – Unknown aperture illumination \( \Rightarrow \) unknown surface
  – Should remain monotone increasing towards a single maximum
  – Iterative solution available
An interferogram is formed by the aperture array
- Each fringe set generated by each aperture pair e.g. \((i,k)\)
- Fringe frequency \(\propto\) spacing between apertures
- Phase and visibility measured by taking Fourier transform

Phase relations of each fringe set/aperture pair can be written as

\[
m_{ik} = \theta_{ik} + \phi_i - \phi_k
\]

and formed into a system of equations rank deficient by \(N\).

Aperture spacings sampling the same spatial frequencies \(\Rightarrow\) same object phase observed
- \((N - 3)\) independent ‘redundant’ conditions necessary for system to be made full rank
  - Array design must satisfy this condition
• Remaining rank deficiency of 3 addressed by a priori object independent information
  - Two a.p. conditions will apply a tilt to the object phases
    • Mislocates object reconstruction, but only morphology important.
  - Final condition sets a reference level for aperture phases.
• Row operations can be performed on resulting matrix to make the system square
  – Zero valued equations removed
  – Corresponding matrix $D$-block is $N$-square and identical to matrix used in image sharpness earlier $\Rightarrow$ same determinant

• Aperture phase estimates depend only on measured phase values $\Rightarrow$ calculated independent of object phases

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
1 & -1 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
\mathit{m}_{34} - \mathit{m}_{12} - \mathit{m}_{13} + \mathit{m}_{24}
\end{bmatrix}
\]

– Object phases $\theta_{ik}$ then calculable to give object reconstruction
Modulo $2\pi$ Arithmetic

- Modulo $2\rho$ nature of phases causes a problem with both methods of correction
- In terms of image sharpness
  - Triangularising the matrix reveals it’s determinant
    - All values will be integers
    - Example here for 4-aperture parallelogram

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
2 & -2 & 0 & 0 \\
0 & 1 & -1 & 0 \\
1 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\hat{\phi}_1 \\
\hat{\phi}_2 \\
\hat{\phi}_3 \\
\hat{\phi}_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
\phi_1 - \phi_2 - \phi_3 + \phi_4 \\
0 \\
0
\end{bmatrix}
\]

\[
\det \begin{bmatrix}
1 & 0 & 0 & 0 \\
2 & -2 & 0 & 0 \\
0 & 1 & -1 & 0 \\
1 & 0 & 0 & -1
\end{bmatrix} = -2
\]

- Appearance of any number $\pm 1$ on the leading diagonal means multiple solutions exist, e.g. with a 2

\[
\phi_1 - \phi_3 + \phi_2 = \phi_2 - \phi_4 - \phi_2
\]
Modulo $2\pi$ Arithmetic

- With phase identification
  - $D$-block identical to the matrix constructed in the image sharpness method
  - Independent system for aperture phases
  - Solving yields the form
    $$\phi_i = \frac{\sum \overline{m}_{lm}}{\det D} = \left(\sum m_{lm} + n2\pi\right) / \det D$$
    - Result may be ambiguous by multiples of $\pi$

- Low determinant (e.g. $\pm 1$) associated with poorer system conditioning
  - Generally
    $$\kappa(A) = \|A^{-1}\| \|A\| = \|\text{adj } A\| \|A\| / \det A$$
RSC Demonstrations

- Image sharpness

- Phase identification
  - Correction of aberrated PSF – note shift due to tilt.
Lucky Imaging

- Filled aperture case
  - Low aberration variance across an aperture sought, but small probability
    - Low variance $\Rightarrow$ Strehl ratio, $S \sim 1$ $\Rightarrow$ relatively undistorted images.
  - Density function of $S$ calculable using Marechal approximation (subject to assumptions).

$$S \approx e^{-\sigma_\phi^2}$$

- Many snapshots necessary to find a ‘lucky’ few.
  - Depends on size of telescope relative to correlation length
  - Image sharpness criterion allows automatic search.
Lucky Imaging

- Synthetic aperture case
  - Aperture array takes $N$ samples of the wavefront
  - Aperture size influences complexity of phase modes passed
    - $\delta$-function samples only piston
    - Real apertures see degrees of tilt and higher modes
    - Typically tilt is the highest significant order
  - Probability of lucky snapshot depends on $N$
    - Joint density function of samples
    - Sufficient frequency content sampled $\Rightarrow$ smaller $N$, higher probability of being lucky
  - Image sharpness solution must be met by the samples
    - Piston only array with redundancies $\Rightarrow$ RSC solution
  - Tilt case more complicated – less chance of being ‘phased’
- L.I. analogous to Free Space Optical Communications
  - Array designs for imaging: low probability of lucky snapshot
  - FSOC aims to optimise array configuration to increase probability
Communications

• Background
  – Free Space Optical Communications using lasers
  – Characteristics of irradiance field vary with time and propagation distance
    • Caustic and speckle formation in the intensity function
  – Large collector required for optimum signal continuity
  – Investigating use of arrays of smaller collectors
    • Airborne platforms requirements
    • Understanding of intensity properties sought
    • Different at different propagation distances
• Theoretical Investigation
  – Analytical approach
    • Based on propagation theory
    • Calculations accurate but unnecessarily unwieldy
  – Phenomenological approach
    • Scattering centres in source field
    • Each point in the target field results from a finite complex summation
    • Probabilistic treatment of scattering centres modelled using ‘bunching’ statistics
    • Closed form expressions for point statistics, e.g. $K$-distribution

\[ \sum \alpha_i e^{j\psi_i} \]
Communications

- Laboratory Experiments
  - Phase functions generated in ±1 orders of detour phase distorted diffraction grating (DDG)
  - Capability
    - Analogue computation of intensity function at almost arbitrary propagation distances by axial positioning of lens L3
  - Limitations
    - Phase screen thin and static
Communications

• Computer Simulations
  – Propagation by angular spectrum of plane waves
  – Capabilities
    • Set of phase functions pseudo-denumerable
    • Dynamic phase functions
    • Ensemble simulations
  – Limitations
    • Thin screen condition remains
    • Single wavelength and limited propagation distance

• Modelling Goals
  – Inductive validation of experimental and computer simulation
  – Validation and qualification of theory
    • Intensity point and spatial statistics
    • Use of theory for FSOC array optimisation
• Validation of lab/computer simulation
  – Left: experimental
  – Right: simulation

Communications

$K$-distribution
(near field)

Negative Exponential
(far field)
2nd Phase Screen

- Modelling propagation through an inhomogeneous medium using a second phase screen
  - OASLM generated phase function propagated onto DDG
    - Up to 1? phase change

- No single phase-only screen can model multiple phase screen propagation
- Allows anisoplanatism to be modelled
- Moving phase patterns possible
2nd Phase Screen

- Intensity statistics relevant to FSOC particularly interesting
  - Rate of emergence and influence of regions distinguishable by intensity function characteristics
    - Focussing/caustics – near field
    - Speckle – far field
  - Influence of the first screen on the second

- Amplitude distribution of the first ‘samples’ regions of the second

- Intensity function due to both is naïve of some regions in the second
Array Design

• Imaging
  – Spatial frequency coverage matching object content important
  – Spacings must include sufficient redundancy for RSC solutions
    • Designs must allow unit determinant
  – Aperture sizes to limit wavefront modes passed

• Communications
  – Array configuration must be optimised to maximise probability of at least one aperture seeing high intensity
    • Intensity characteristics depend on atmosphere and propagation distance
  – Platform limits aperture sizes
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