Phase Retrieval with the Intensity Transport Equation: Orthogonal Series Solution for Non-uniform Illumination

Sijiong Zhang

Physics, School of Engineering and Physical Sciences, Heriot-Watt University, Edinburgh, Scotland, UK EH14 4AS

References

1. T. Nugent, J. Opt. Soc. Am. A/ Vol. 13, No. 8/August 1996, 1670-1682
2. T. Nugent, J. Opt. Soc. Am. A/ Vol. 12, No. 9/Sept. 1995, 1932-1941
3. A. Greenaway, H. Campbell, 4th INTERNATINAL WORKSHOP ON ADAPTIVE OPTICS FOR INDUSTRY AND MEDICINE

Outline

1. The intensity transport equation(ITE) in the case of non-uniform intensity.

2.The method for phase retrieval by the ITE with use of Zernike polynomials.

3. Phase retrieval results for ATC data

Intensity Transport Equation(ITE)

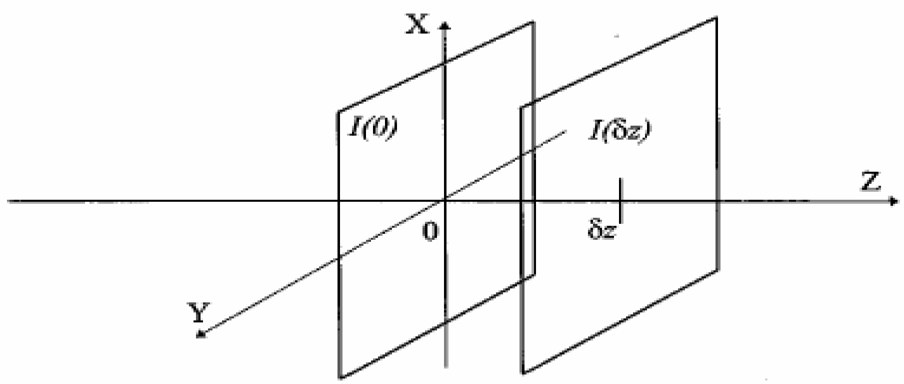

Fig. 1. Phase retrieval can be performed by using the two intensity measurements $I(x, y, 0)$ and $I(x, y, \delta z)$ on adjacent planes $z = 0$ and $z = \delta z$ orthogonal to the optical axis z.

$$k\partial_z I = -\nabla \cdot (I\nabla\varphi), \qquad (1)$$

Helmholtz's Equation
Paraxial approximation

where $\nabla = (\partial_x, \partial_y)$ is the gradient operator in the plane

$\partial_z = \partial/\partial z$, $k = 2\pi/\lambda$ is the wave number

Assumptions for solving ITE

I(x,y) is the smooth function

$$I(x, y) > 0 \quad \text{inside } \Omega, \qquad (2)$$

$$I(x, y) \equiv 0 \quad \text{outside } \Omega \text{ and on } \Gamma. \qquad (3)$$

The intensity in the area Ω of the image is positive and the intensity vanishes outside Ω.

Thus to retrieve the phase φ we must solve Eq. (1) with *I*(*x*,*y*) satisfying relations (2) and (3) in area Ω.

The solution to problem (1)–(3) is always unique up to an arbitrary additive constant.

Necessary Condition for Solution Existence

For the phase solution to problem(1)–(3) to exist, the following condition must hold:

$$\int\int_{\Omega} \partial_x I \mathrm{d}x \mathrm{d}y = 0. \qquad (4)$$

This condition means the energy conservation law and can be used to check the consistency of experimentally measured intensity data.

Solving the Intensity Transport Equation using Zernike polynomials

The Zernike polynomials $Z_j(r/R,\theta)$, $0 \leqslant r \leqslant R$, $j = 1, 2, 3, \ldots$, make up a complete orthonormal set in Ω with respect to the scalar product:

$$\langle f, g \rangle = R^{-2} \int_0^{2\pi} \int_0^R f(r,\theta) g(r,\theta) r \,\mathrm{d}r \,\mathrm{d}\theta. \qquad (5)$$

Let us multiply ITE (1) both sides by the Zernike polynomial *Zj*(*r*/*R*, θ) and integrate it over Ω :

$$-R^{-2} \int_0^{2\pi} \int_0^R \nabla \cdot (I \nabla \varphi) Z_j r \,\mathrm{d}r \,\mathrm{d}\theta = R^{-2} \int_0^{2\pi} \int_0^R F Z_j r \,\mathrm{d}r \,\mathrm{d}\theta, \qquad (6)$$

where $F = k\partial_z I(r,\theta)$.

Solving the Intensity Transport Equation using Zernike polynomials

The right-hand side of Eq. (6) is by definition the *j*th Zernike coefficient $F_j = \langle F, Z_j \rangle$ of the function *F*.

On the left-hand side of Eq. (6) we decompose φ into Zernike terms,

$$\varphi(r,\theta) = \sum_{i=1}^{\infty} \varphi_i Z_i(r/R,\theta) \qquad (7)$$

and integrate Eq.(6) by parts, taking Eq. (3) into account. The integral over the boundary Γ disappears, and we obtain

$$\sum_{i=1}^{\infty} \varphi_i R^{-2} \int_0^{2\pi} \int_0^R I \nabla Z_i \cdot \nabla Z_j r \, \mathrm{d}r \, \mathrm{d}\theta = F_j . \qquad (8)$$

Now it is convenient to introduce the matrix $\mathbf{M} = [M_{ij}]$ with elements

$$M_{ij} = \int_0^{2\pi} \int_0^R I(r,\theta) \nabla Z_i(r/R,\theta) \cdot \nabla Z_j(r/R,\theta) r \mathrm{d}r \mathrm{d}\theta, \quad i, j = 1, 2, 3, \ldots . \qquad (9)$$

Solution of the Intensity Transport Equation With Non-uniform Intensity

Using this definition we can rewrite Eq. (8) as a system of algebraic equations for the unknown Zernike coefficients of the phase: φ_i

$$\sum_{i=1}^{\infty} M_{ij}\varphi_i = R^2 F_j, \quad j = 1, 2, 3, \ldots, \quad \text{or } \mathbf{M}\varphi = R^2 F.$$

Further we define

$$\delta_z I_{(N)} = \int_0^{2\pi}\int_0^{R} [I(r, \theta, \delta z) - I(r, \theta, 0)] Z_j r \,\mathrm{d}r\,\mathrm{d}\theta. \qquad (10)$$

$$N_F = \frac{2\pi R^2}{\lambda \delta z}$$ (the dimensionless scaling factor)

So for truncated Zernike polynomials, the retrieval phase coefficients can be written as

$$\varphi_{(N)} = N_F \mathbf{M}_{(N)}^{-1} \delta_z I_{(N)}, \qquad (11)$$

Key result

Condition for using ITE to retrieve phase

`Null Sensor` : The difference between two aberrated diversity images is null if the input wave-front is plane wave and is non-null for non-plane wave-fronts.

Because ITE includes the difference between two aberrated diversity images(Eq.(10)), this method just extracts the non-plane wave-fronts.

The condition that the above result is valid is two phase diversities must be same value but opposite sign.

Result